# Identifiability of Subsampled/Mixed-Frequency Structural VAR models

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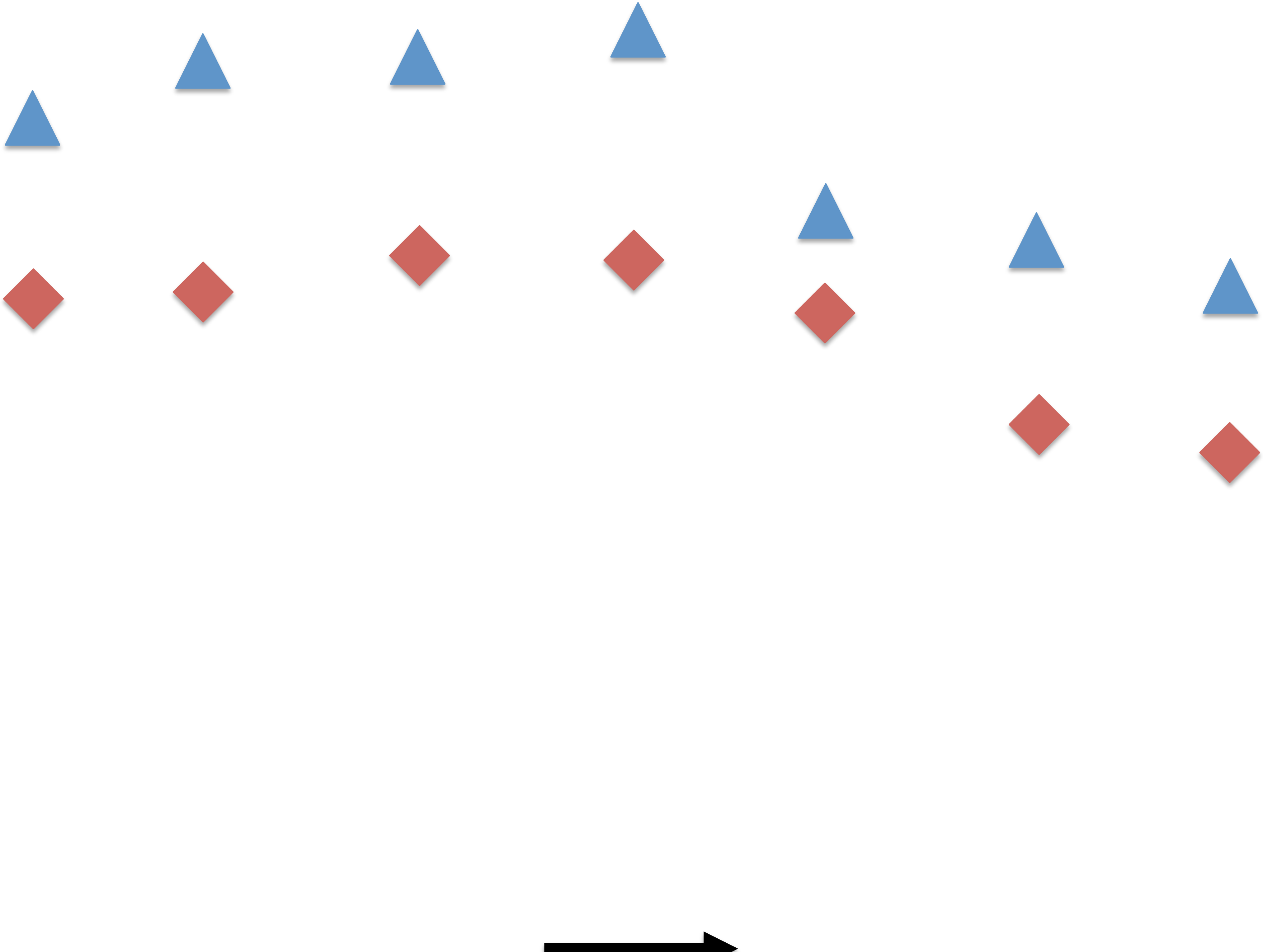
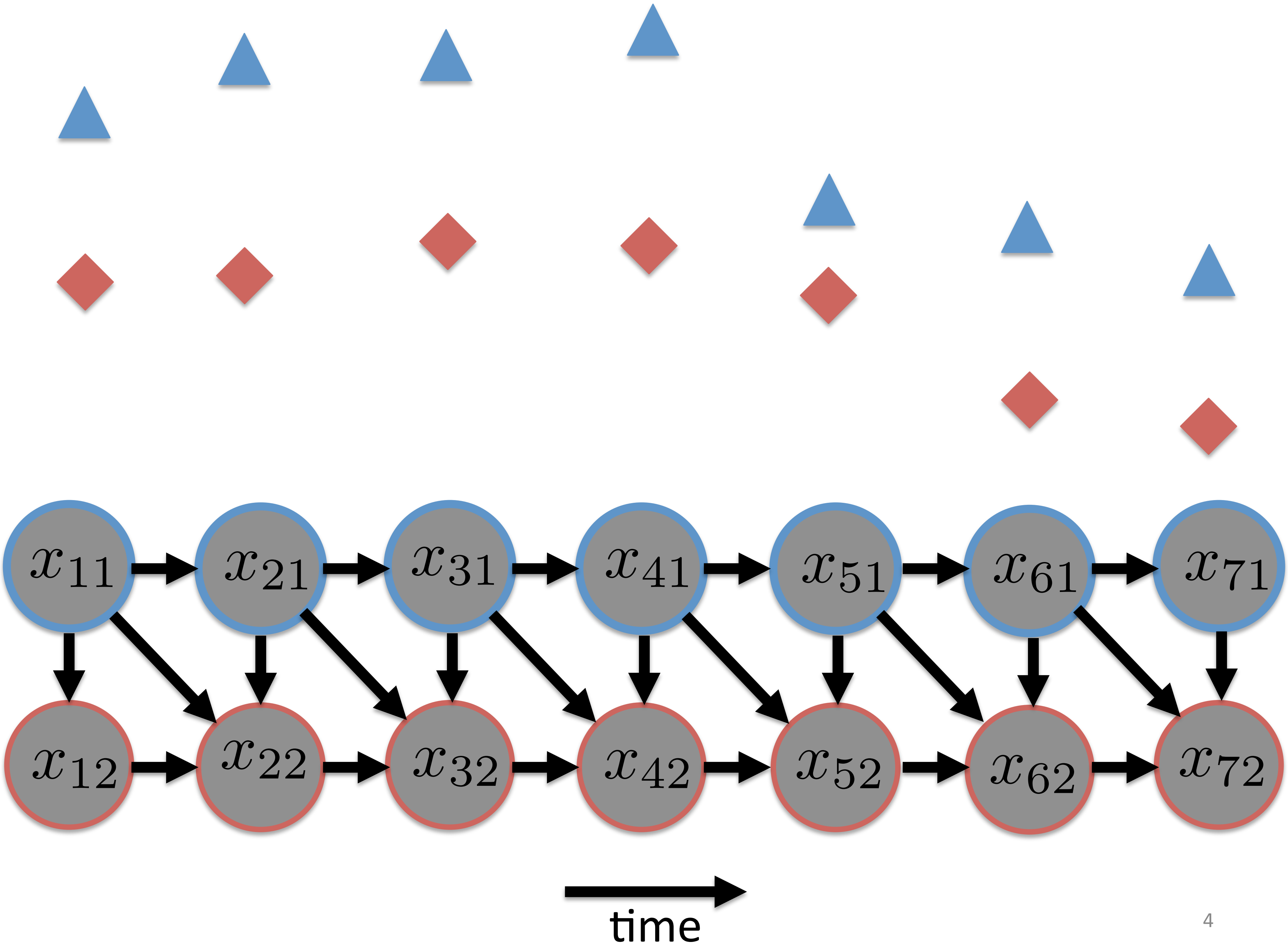
Joint work with Emily Fox and Ali Shojaie


time


time


$x_{11}$
$x_{21}$
$x_{31}$
$x_{41}$
$x_{51}$
$x_{61}$
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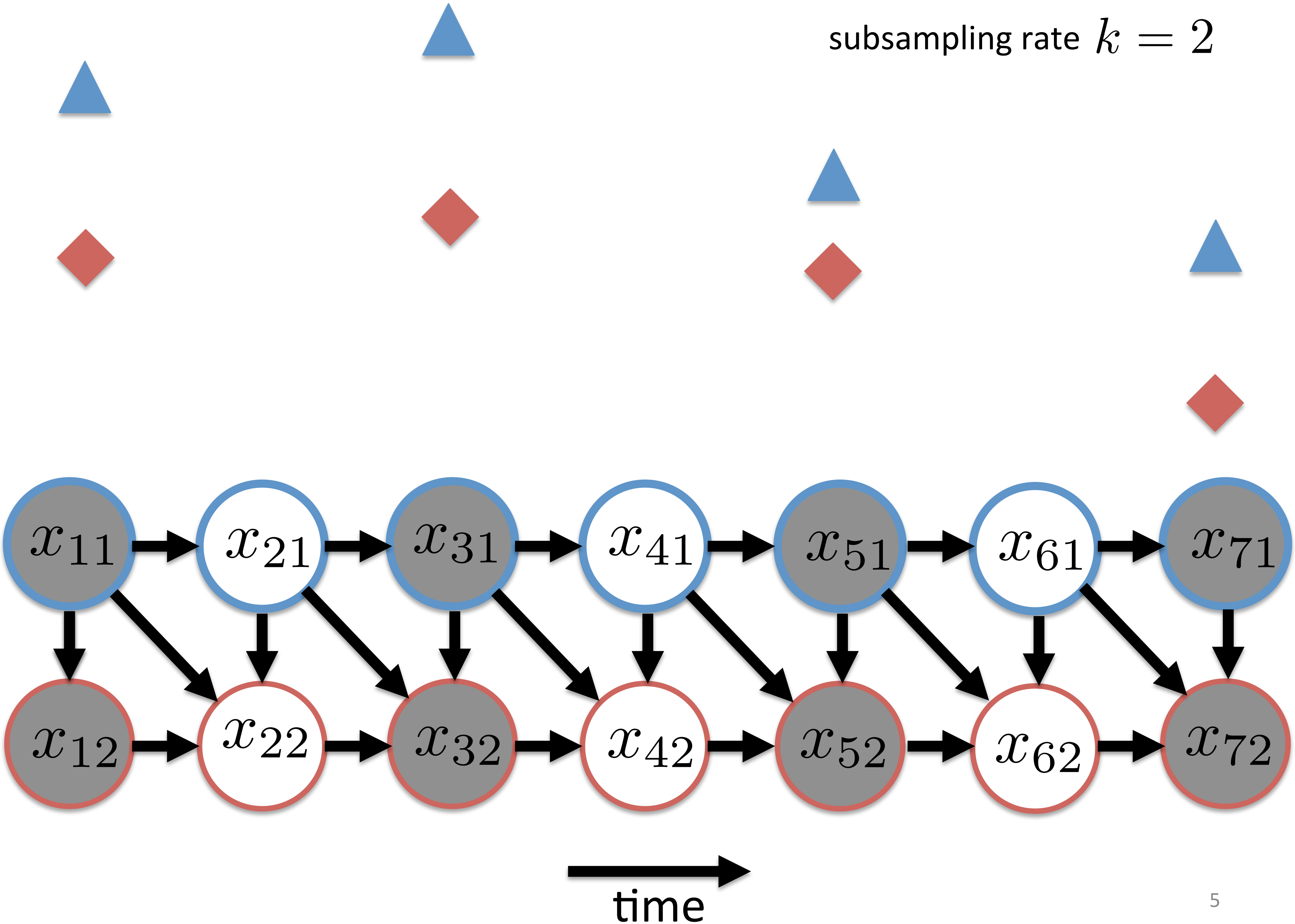
$x_{72}$
time


subsampling rate $k = 2$
$x_{11}$
$x_{21}$
$x_{31}$
$x_{41}$
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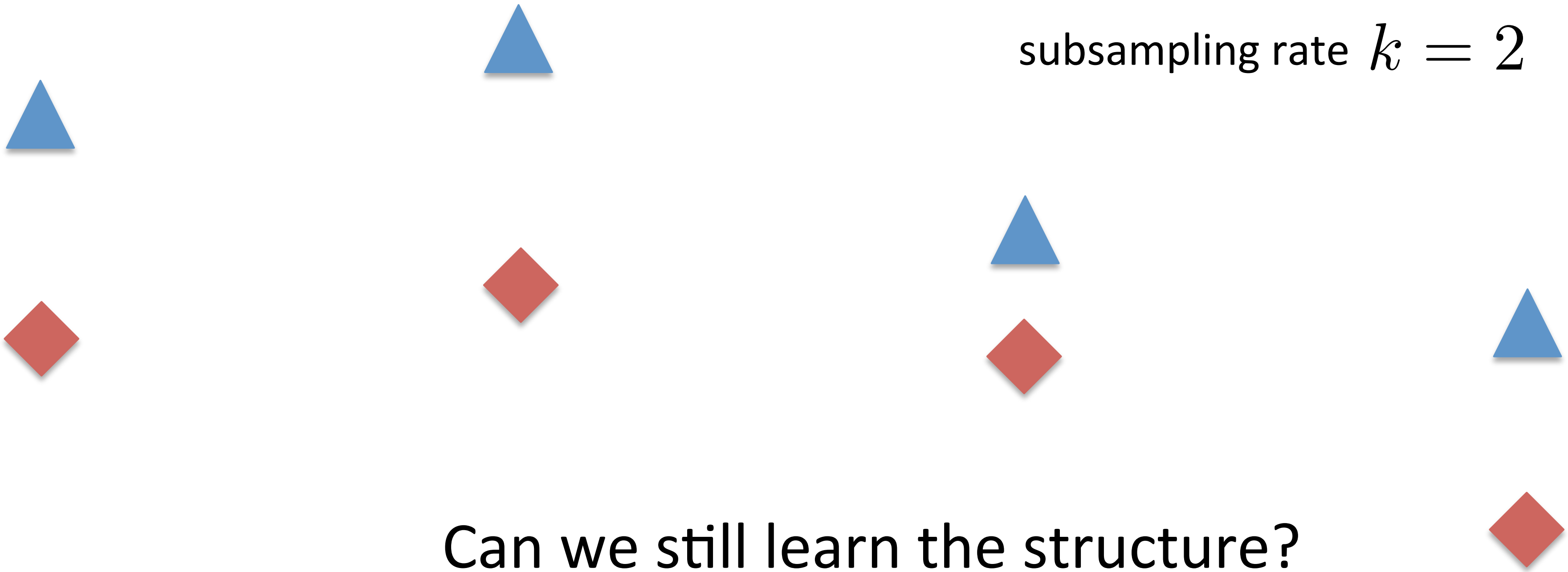
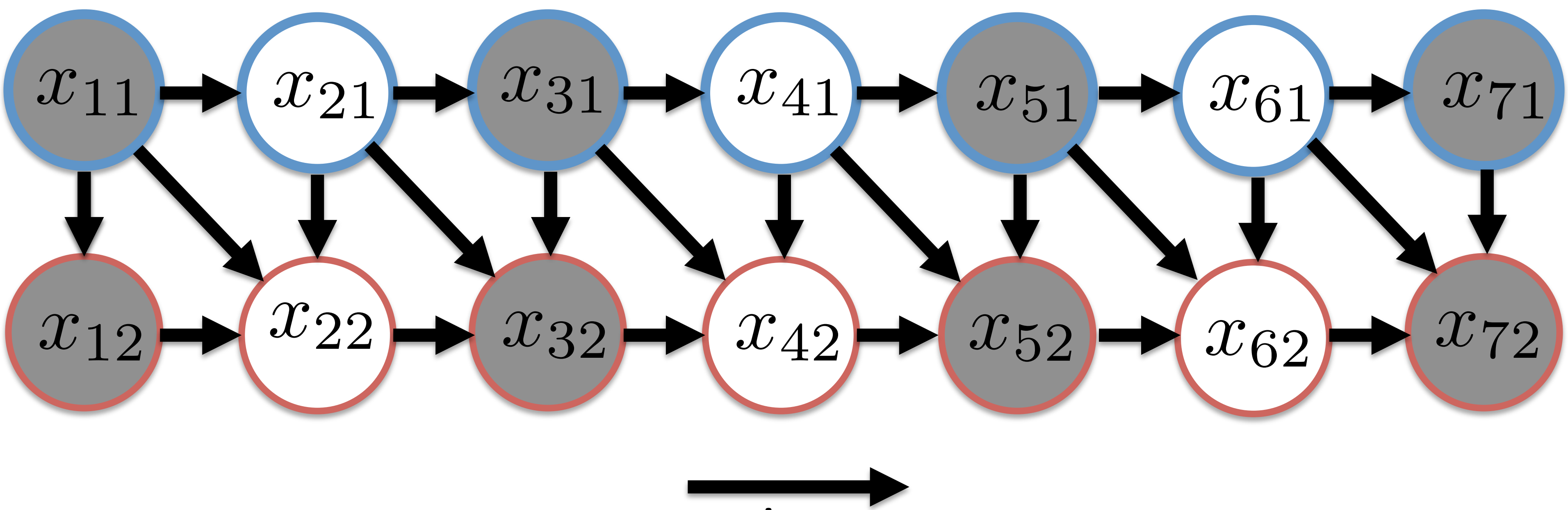
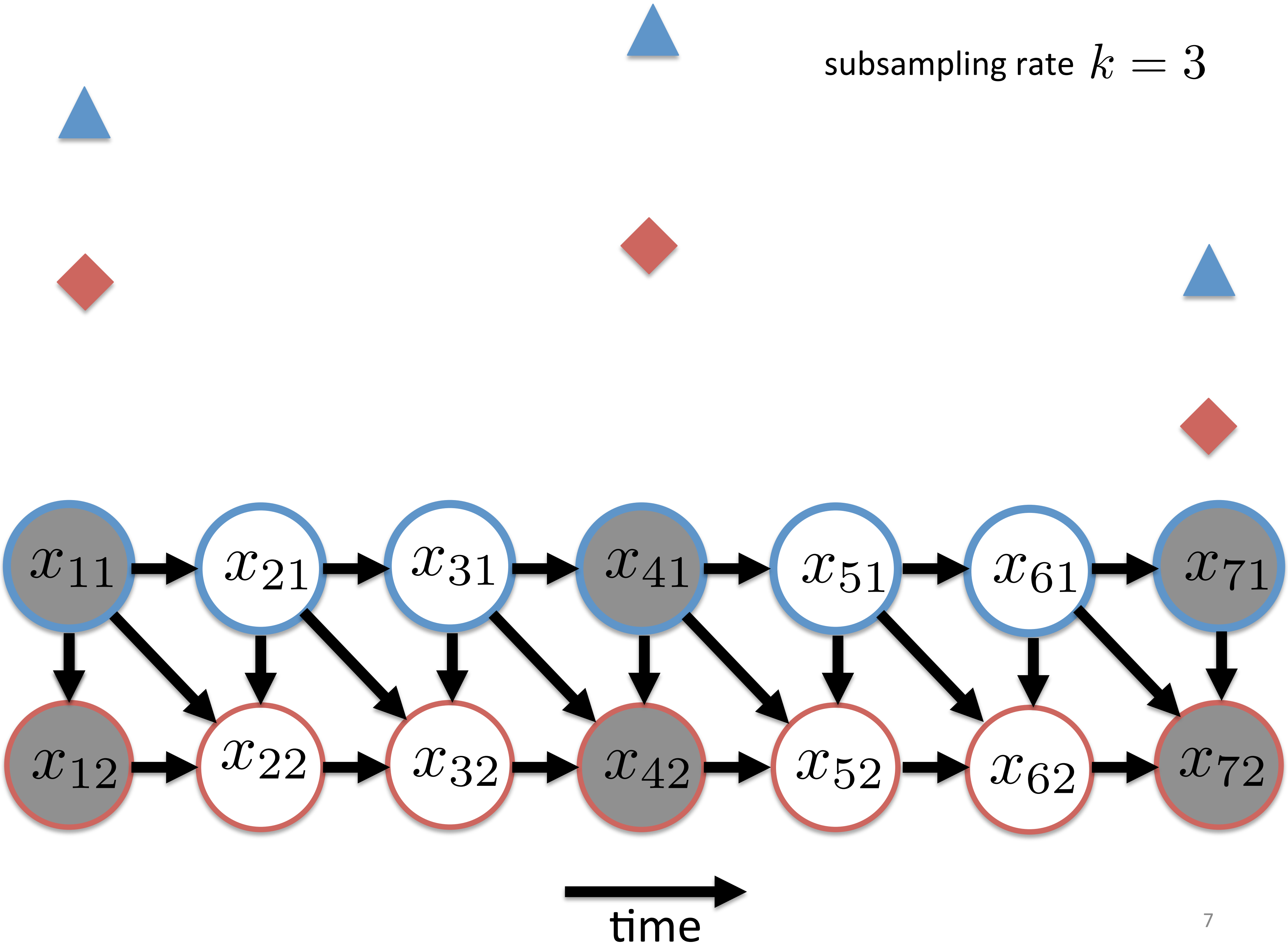


Can we still learn the structure?




subsampling rate $k = 3$
$x_{11}$
$x_{21}$
$x_{31}$
$x_{41}$
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$x_{12}$
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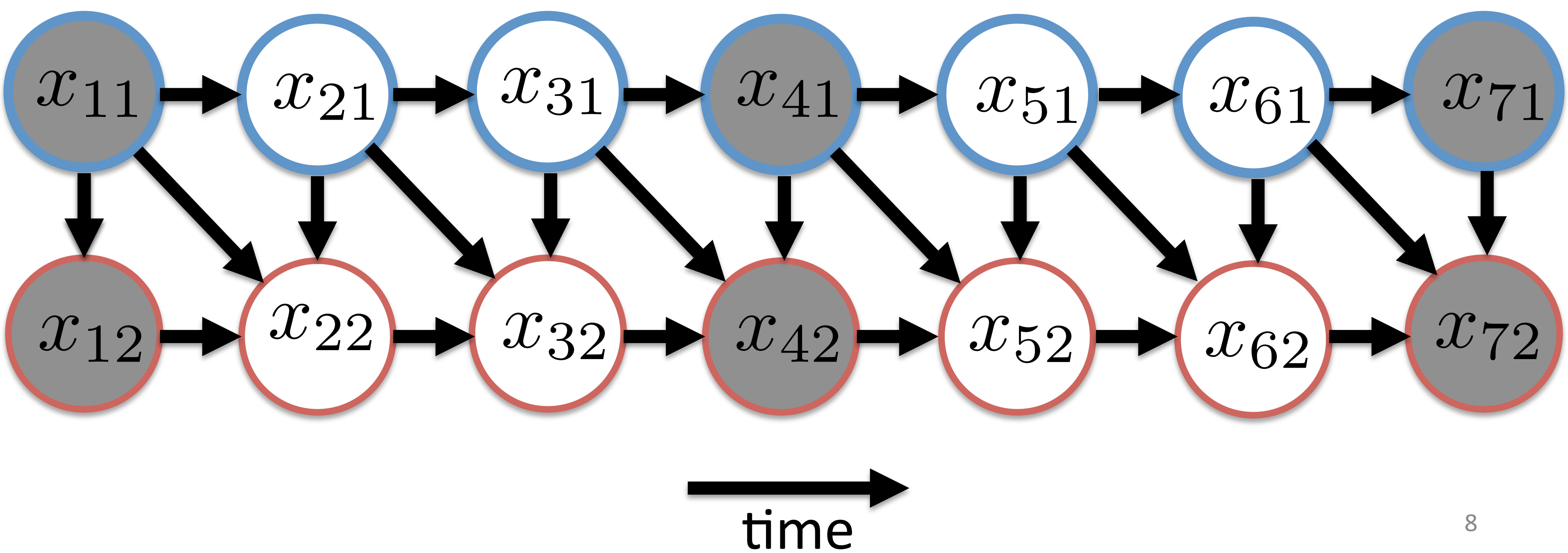
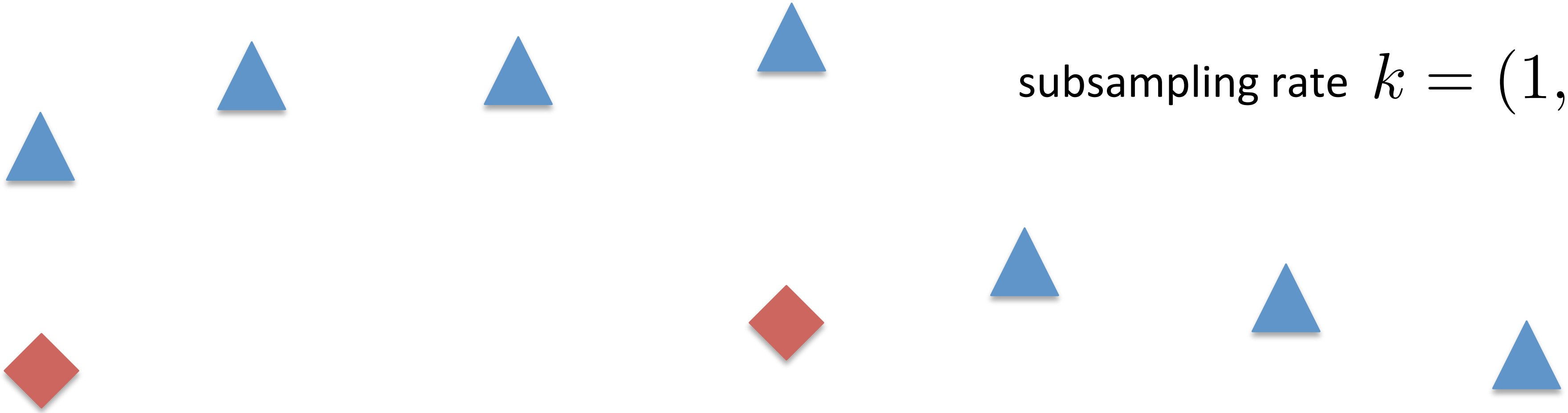
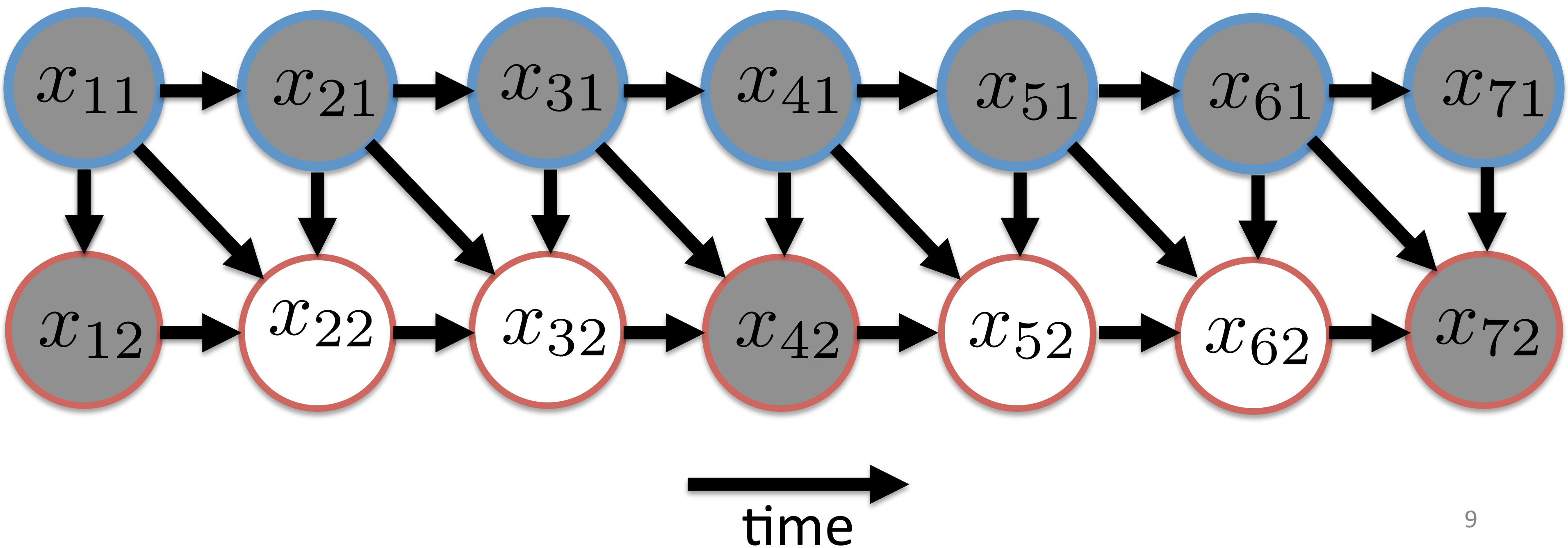


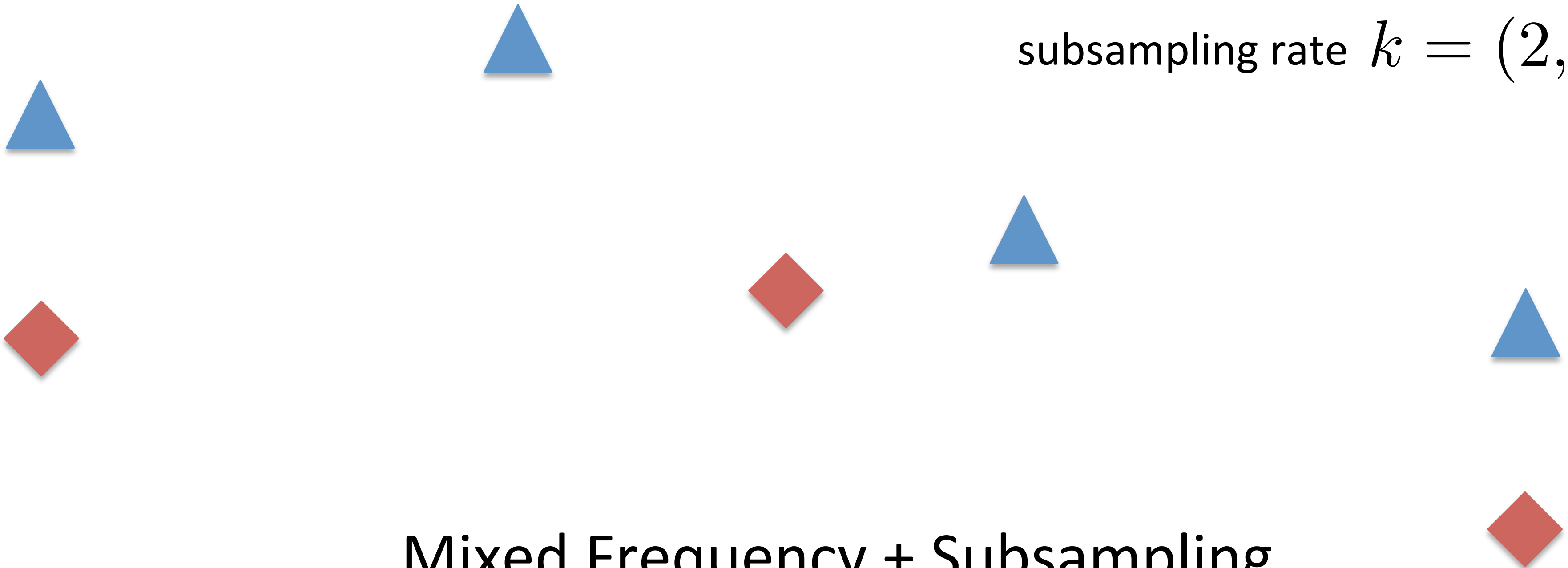
subsampling rate $k = (1, 2)$



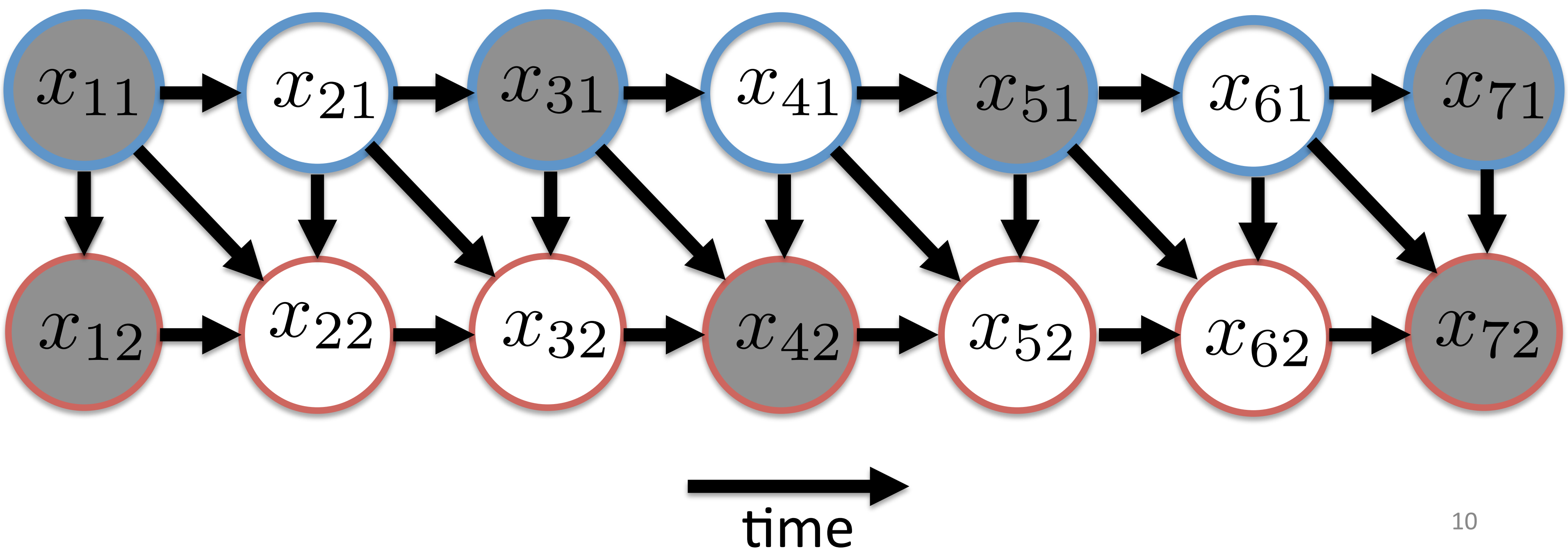
## Mixed Frequency (MF)



subsampling rate $k = (2, 3)$



## Mixed Frequency + Subsampling



# Causes of Subsampling and Mixed Frequencies

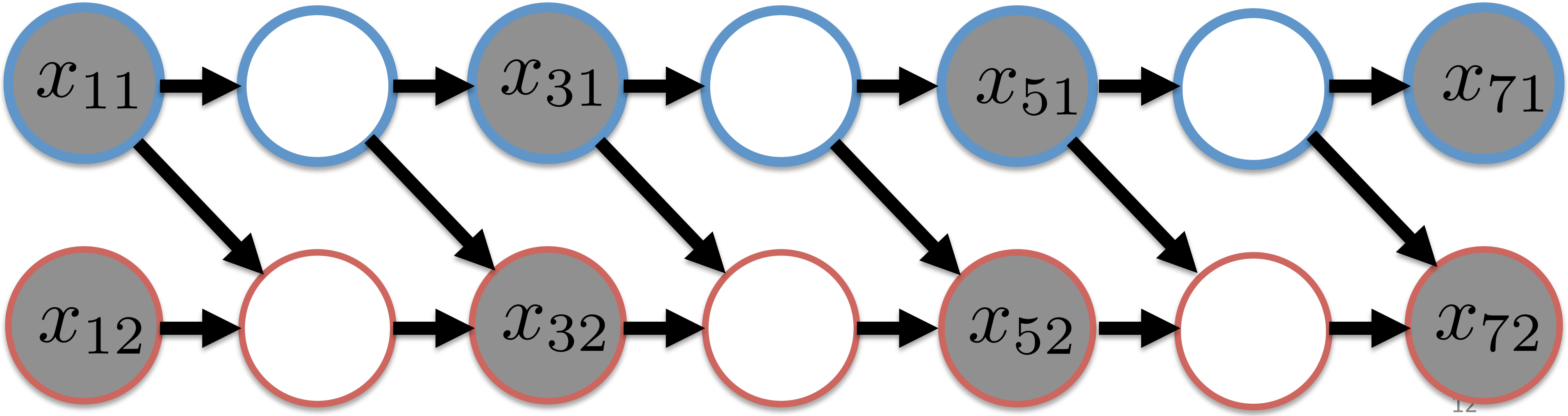
- Costly data collection:
  - GDP
  - Housing prices
  - Other econometric indicators.
  - Biomarker health indicators.

- Technological limitations:
  - fMRI/EEG all sample neural activity at fixed rates.

# Previous Work

- Gong et al. 2015 study subsampled VAR models with independent errors.

$$x_t = \mathbf{A}x_{t-1} + e_t$$

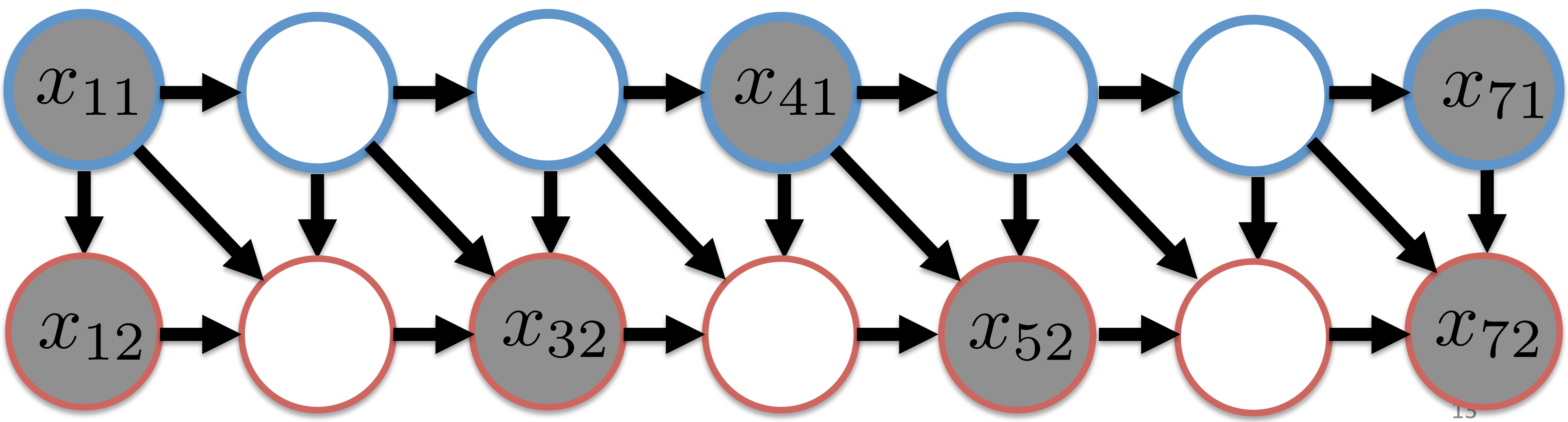


# Previous Work

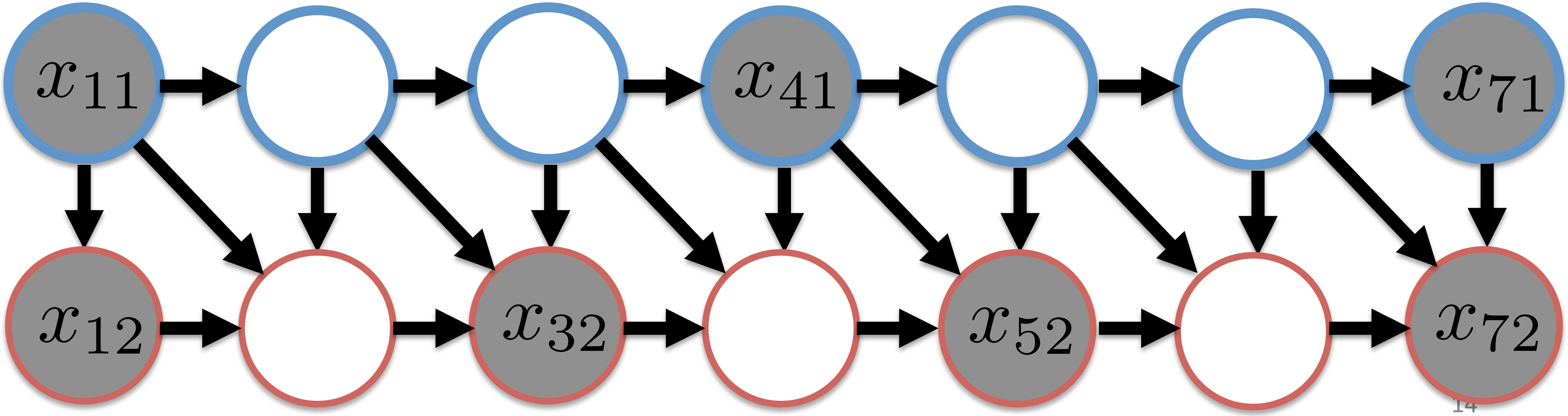
- Gong et al. 2015 study subsampled VAR models with independent errors.

$$x_t = \mathbf{A}x_{t-1} + e_t$$

- We extend their framework to deal with mixed subsampling frequencies and correlated errors.

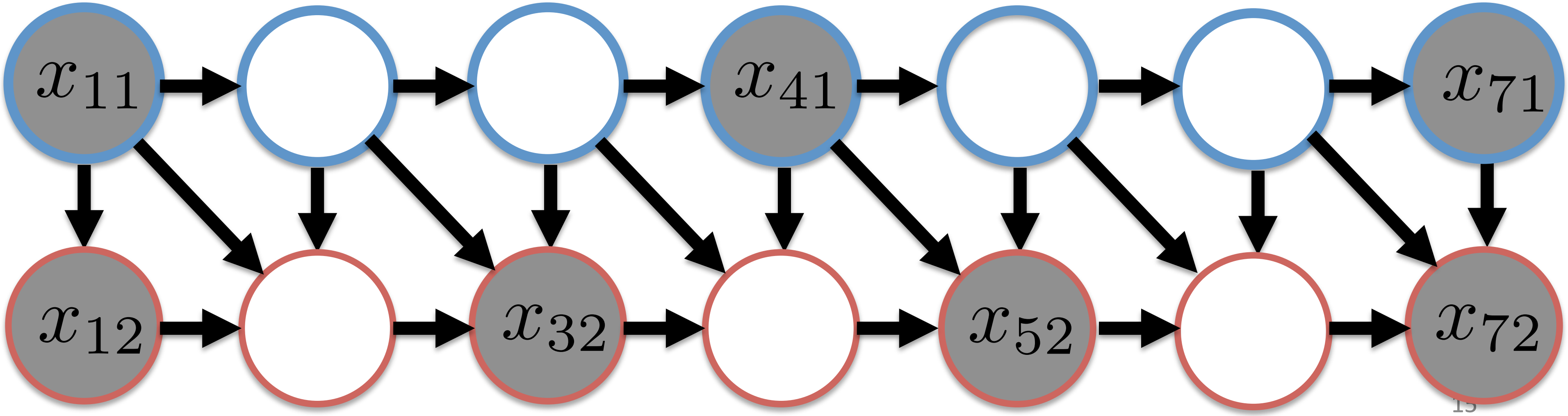


# Structural Vector Autoregressive Model (SVAR)



# Structural Vector Autoregressive Model (SVAR)

$$x_t = \mathbf{A}x_{t-1} + \mathbf{C}e_t$$



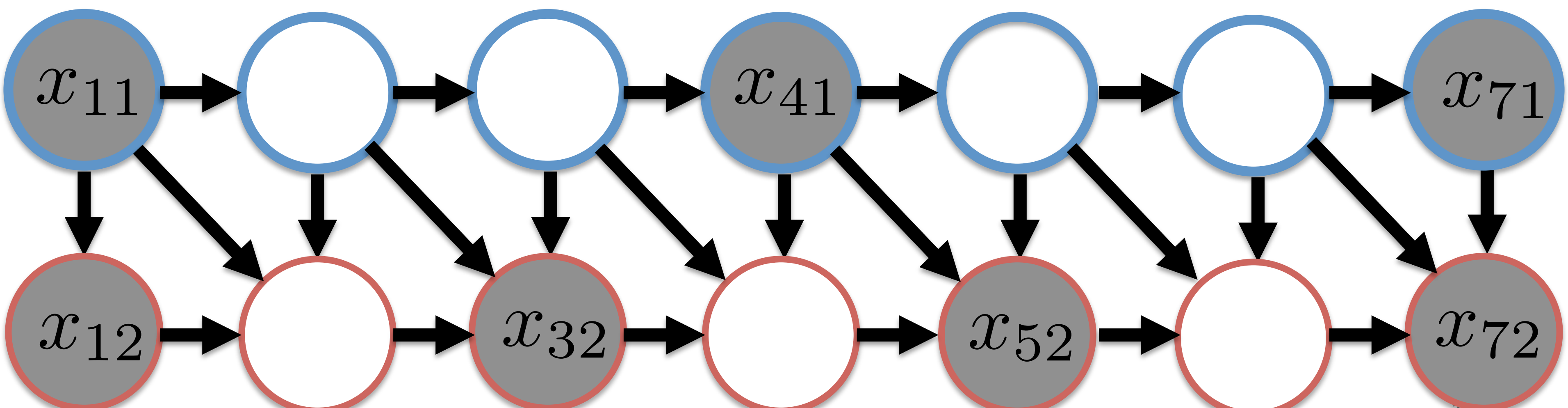
# Structural Vector Autoregressive Model (SVAR)

$$x_t = \mathbf{A} x_{t-1} + \mathbf{C} e_t$$

Instantaneous errors, 'shocks'

transition matrix

structural matrix



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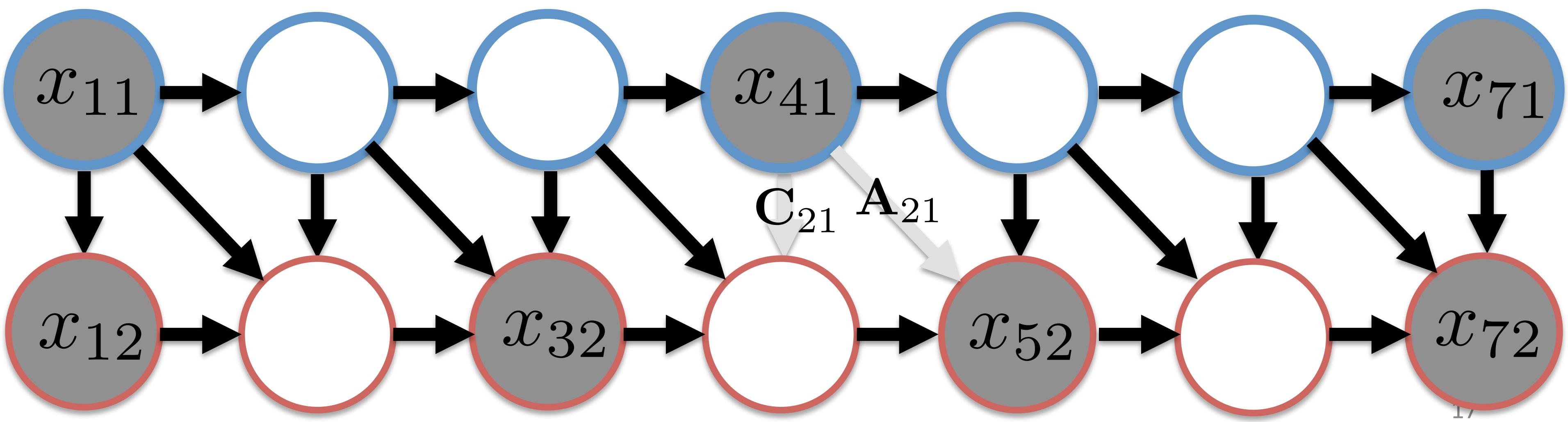
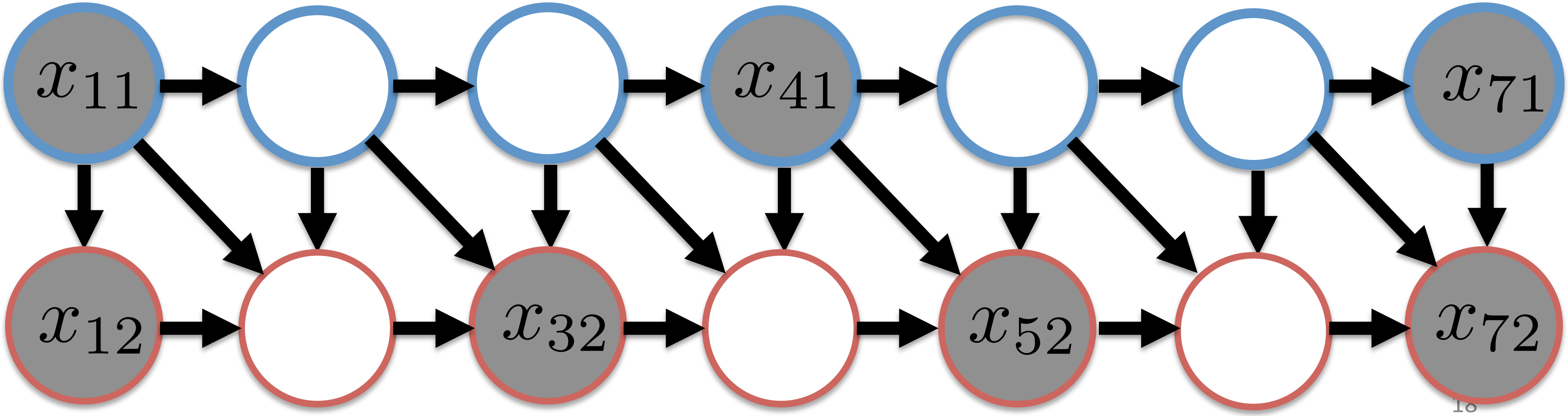


# Subsampled/MF SVAR

$$x_t = \mathbf{A}x_{t-1} + \mathbf{C}e_t$$



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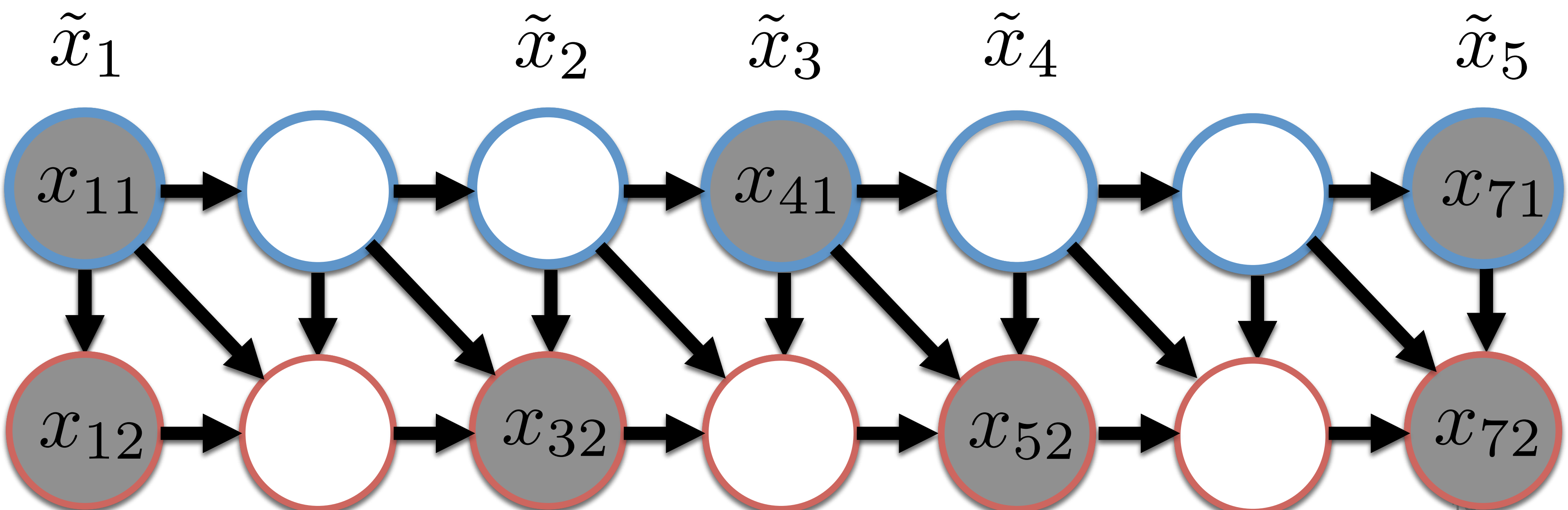
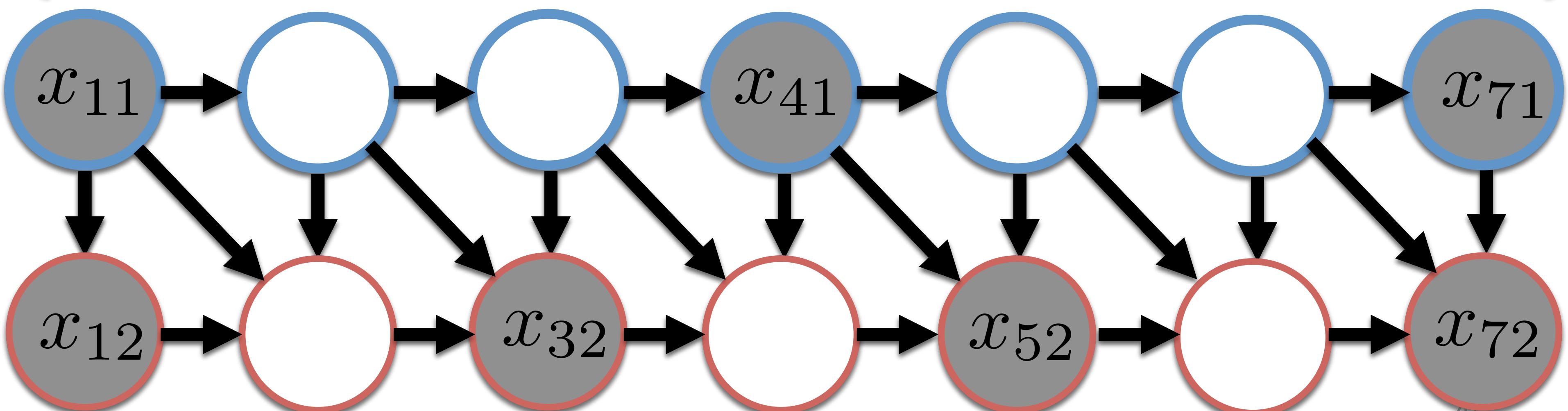


# Subsampled/MF SVAR

$$x_t = \mathbf{A}x_{t-1} + \mathbf{C}e_t$$

$$\tilde{\mathbf{X}} = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 & \tilde{x}_5 \end{bmatrix}$$



# Subsampled SVAR process

- When subsampling at same rate:

$$\tilde{x}_t = \mathbf{A}^k \tilde{x}_{t-1} + \mathbf{L}\tilde{e}_t$$

$$\mathbf{L} = \left(\mathbf{C}, \mathbf{AC}, \ldots, \mathbf{A}^{k-1}\mathbf{C}\right)$$

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- $(\mathbf{A}, \mathbf{C})$ not identifiable from first two moments of $\tilde{\mathbf{X}}$ !
  - Implies not identifiable if $e_t$ Gaussian.

# Non-Gaussian SVAR

$$x_t = \mathbf{A}x_{t-1} + \mathbf{C}e_t$$

| | | |
|---|---|---|
| $e_t$ | **A1** | $e_{it}$ independent of $e_{jt}$ |
| | **A2** | $e_{it} \sim p_{e_i}$ non-Gaussian |
| | **A3** | $p_{e_i} \neq p_{e_j} \quad \forall i, j$ |
| $\mathbf{C}$ | | |

# Non-Gaussian SVAR

$$x_t = \mathbf{A} x_{t-1} + \mathbf{C} e_t$$

| | |
|---|---|
| $e_t$ | **A1** $e_{it}$ independent of $e_{jt}$<br>**A2** $e_{it} \sim p_{e_i}$ non-Gaussian<br>**A3** $p_{e_i} \neq p_{e_j} \quad \forall i, j$ |
| $\mathbf{C}$ | No restrictions ⟶ $\mathbf{C}$ identifiable<br>(Lanne et al 2015)<br><br>Perm. to lower triangular ⟶ DAG and its ordering associated w/ $\mathbf{C}$ identifiable.<br>(Hyvarninen et al 2010, 2013 and Peters et al 2013 ) |

# Subsampled/MF Non-Gaussian SVAR

- $(\mathbf{A}, \mathbf{C}, e, k)$ Non-Gaussian subsampled/MF parameterization

- Identifiability in this setting is a unique map between

$$\text{Distribution of } \tilde{\mathbf{X}} \longleftrightarrow (\mathbf{A}, \mathbf{C}, e, k)$$

- Proof technique:
  - Show that if two parameterizations $(\mathbf{A}, \mathbf{C}, e, k)$ and $(\mathbf{A}', \mathbf{C}', e', k)$ lead to same distribution of $\tilde{\mathbf{X}}$ then $(\mathbf{A}, \mathbf{C}) = (\mathbf{A}', \mathbf{C}')$.

# Identifiability for Subsampled/MF SVAR

**Theorem**: *Suppose* $\tilde{\mathbf{X}}$ *is generated according to subsampled/MF SVAR process* $(\mathbf{A}, \mathbf{C}, e, k)$ *and also admits another representation* $(\mathbf{A}', \mathbf{C}', e', k)$. *Assume A1-3 hold and* $||\mathbf{A}||_2 < 1$

$\mathbf{C} = \mathbf{C}'P$ *where* $P$ *is a permutation matrix with 1 and -1 entries.*

*If* $\mathbf{C}$ *lower triangular* $\rightarrow$ $\mathbf{C} = \mathbf{C}'$

*If* $p_{e_i}$ *asymmetric and* $\mathbf{C}$ *full rank* $\rightarrow$ $(\mathbf{A}, \mathbf{C}) = (\mathbf{A}', \mathbf{C}')$

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**Corollary**: *If the instantaneous interactions follow a DAG structure,* $\mathbf{C}$ *may be permuted to lower triangular, DAG structure and ordering identifiable from* $\tilde{\mathbf{X}}$.

# Bayesian Estimation: Gibbs Sampler

Non-Gaussianity:

- Model the $e_{it}$ as a mixture of Gaussians with 2 components.
- Introduce auxiliary $z_{it}$ binary variables such that:

$$e_{it} = z_{it} w_{it}^1 + (1 - z_{it}) w_{it}^2$$

$$w_{it}^1 \sim \mathcal{N}(\mu_i^1, \tau_i^1) \qquad w_{it}^2 \sim \mathcal{N}(\mu_i^2, \tau_i^2)$$

Subsampling:

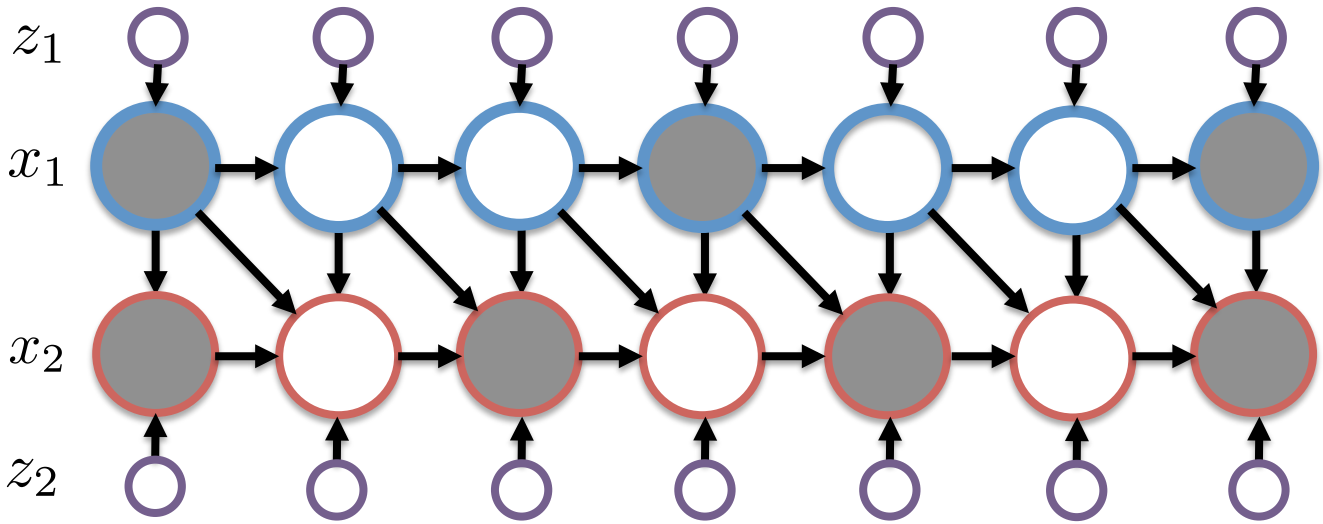
- Treat the unobserved data in $\mathbf{X}$ as *missing*. Sampler will impute missing values.

# Bayesian Estimation: Gibbs Sampler

Place conjugate priors on all parameters $\Theta = (\mathbf{A}, \mathbf{C}, \mu, \tau, \pi)$

Gibbs sampler steps:

1. Jointly impute missing data: $(\mathbf{X}, \mathbf{Z}) \sim p(\mathbf{X}, \mathbf{Z} | \tilde{\mathbf{X}}, \Theta)$

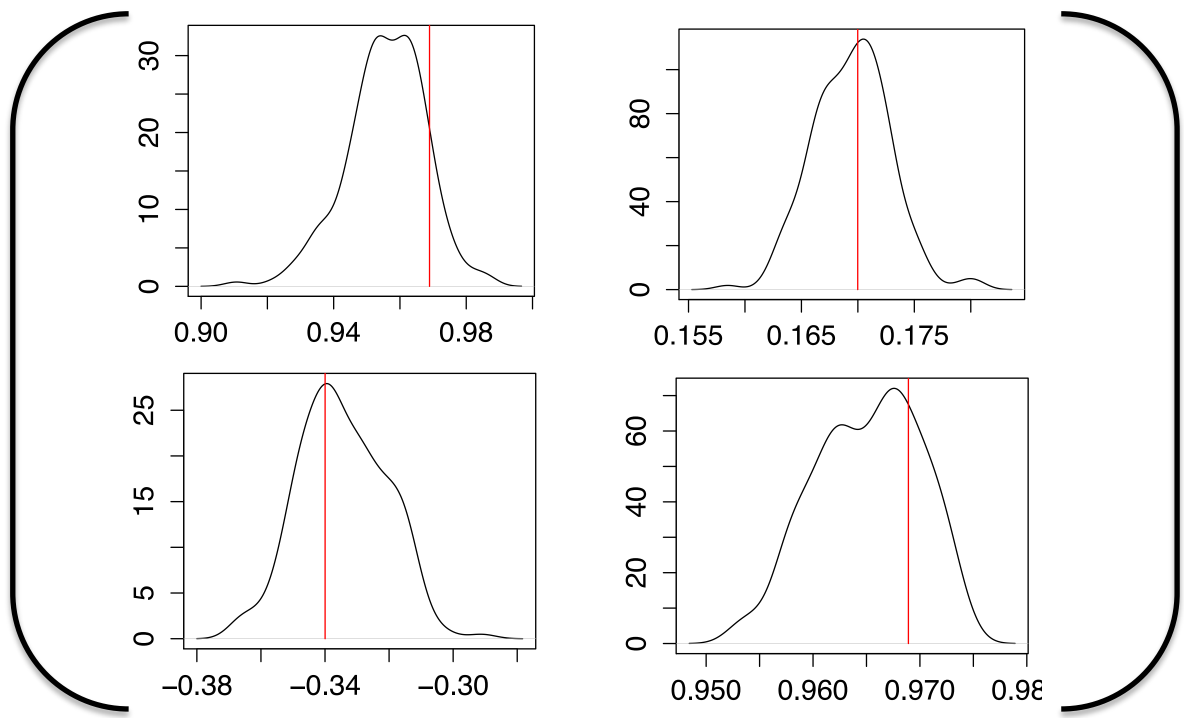


2. Standard conditional Gibbs updates for all $\Theta$.
    1. Sample $\mathbf{C}$ as in Wozniak et al 2015.

# Bayesian Estimation: Gibbs Sampler

- Simulation with subsampling rate k = 2.
- C = I, T = 403

$$p(\mathbf{A}|\tilde{\mathbf{X}}) \approx$$



# Conclusion

- Extended identifiability results of Gong et al 2015 to structural VAR models with mixed subsampling.
- Allows instantaneous covariance and interactions.
- May identify the structural matrix and transition matrix under non-Gaussian errors.
- Developed Gibbs sampler for inference.

# References


- Gong et al. 'Learning Temporal Causal Relations from Subsampled Time Series' ICML 2015
- Wozniak et al. 'Assessing Monetary Policy Models: Bayesian Inference for Heteroskedastic Structural VARs' 2015